we know it's invertible if in REF and no row of all 0's

get into reduced echelon form

a & d always equiv, b & e

A is invertible

$Ax=b$

$x = A^{-1}\vec{b}$

10/30 Lecture Notes

Ex) compute: $\begin{bmatrix} 2 & 5 & 0 & 0 \\ 1 & 3 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -2 \end{bmatrix}^{-1} \rightarrow [A|I]$

$$\left[\begin{array}{cccc|cccc} 2 & 5 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -2 & 0 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cccc|cccc} 1 & \frac{5}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -2 & 0 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cccc|cccc} 1 & \frac{5}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -2 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{cccc|cccc} 1 & 3 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & -2 & 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & -1 & 1 \end{array}\right] \sim \left[\begin{array}{cccc|cccc} 1 & 3 & 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & -1 & 0 & 0 & 1 & -2 & 4 & -2 \\ 0 & 0 & 1 & 0 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & -1 \end{array}\right] \sim \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 3 & -5 & 10 & -5 \\ 0 & 1 & 0 & 0 & -1 & 2 & -4 & 2 \\ 0 & 0 & 1 & 0 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & -1 \end{array}\right]$$

Therefore, $A^{-1} = \begin{bmatrix} 3 & -5 & 10 & -5 \\ -1 & 2 & -4 & 2 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$

Big Theorem (V.3):

Let $\mathcal{A} = \{a_1, \dots, a_n\}$ be a set of n vectors in $\mathbb{R}^n$; let $A = [a_1, \dots, a_n]$ and let $T: \mathbb{R}^n \to \mathbb{R}^n$ be given by $T(\vec{x}) = A\vec{x}$. Then the following are equivalent:

a) $\mathcal{A}$ spans $\mathbb{R}^n$

b) $\mathcal{A}$ is linearly independent

c) $Ax=b$ has a unique solution for all b in $\mathbb{R}^n$

d) T is onto

e) T is 1-1

f) A is invertible

a & d are always equivalent; b & e are always equivalent

Subspaces

- vectors making a plane in $\mathbb{R}^3 \neq \mathbb{R}^2$, e.g.
  - ↳ That plane is a subspace
- Definition: A subset of $\mathbb{R}^n$ is called a subspace if S satisfies the following conditions:
  - a) $\vec{0}$ is in S $(0 \in S)$ } means set can't be empty
  - b) If $\vec{u}$ and $\vec{v}$ are in S then $\vec{u}+\vec{v}$ is in S
  - c) If $\vec{u}$ is in S & c is a scalar, then $c\vec{u}$ is in S

Ex) Let S be the set of vectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ with $x_1 + x_2 = 0$. Is S a subspace?

a) $\vec{0}$ is in the set because $0+0=0 \Rightarrow$ a) ✓

b) $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \Rightarrow \vec{u}+\vec{v} = \begin{bmatrix} u_1+v_1 \\ u_2+v_2 \end{bmatrix}$ then $(u_1+v_1)+(u_2+v_2) \rightarrow \underbrace{\overset{0}{u_1+u_2}+\overset{0}{v_1+v_2}}_{0+0=0}$

so $0+0=0 \Rightarrow$ b) ✓

c) $cu_1 + cu_2 = 0 \Rightarrow c\overset{0}{(u_1+u_2)} = 0 \Rightarrow cu$ is in $S \Rightarrow$ c) ✓

Therefore S must be a subspace (the line $y=-x$ in $\mathbb{R}^2$)

If changed to $x_1 + x_2 = 1$, is it still a subspace? No, because it no longer passes a) or passes through the origin

10/30 Lecture Notes (continued)

Theorem: Let $S = \text{span}(\vec{u}_1, \dots, \vec{u}_m)$ be a subset of $\mathbb{R}^n$. Then S is a subspace

Proof. a) $\vec{0} = 0\vec{u}_1 + \dots + 0\vec{u}_m$ so $\vec{0}$ is in S

b) let u and v in S so $\vec{u} = c_1u_1 + \dots + c_mu_m$ and $\vec{v} = d_1u_1 + \dots + d_mu_m$

thus $\vec{u}+\vec{v} = (c_1+d_1)u_1 + \dots + (c_m+d_m)u_m$ so $u+v$ is in S

c) $cu = cc_1u_1 + \dots + cc_mu_m$ so cu is in S thus S is a subspace

Ex) Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & -2 \end{bmatrix}$ let S be the set of solutions to $A\vec{x} = \vec{0}$. Is S a subspace?

$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 2 & -1 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & -1 & -4 & 0 \end{array}\right]$

$x_3 = \text{free variable} = s_1$

$x_2 = -4s_1$

$x_1 = -s_1$

$\vec{x} = s_1 \begin{bmatrix} -1 \\ -4 \\ 1 \end{bmatrix}$

$S = \text{span}\left(\begin{bmatrix} -1 \\ -4 \\ 1 \end{bmatrix}\right)$

(the set of solutions to $Ax = 0$ is a subspace as well)

Theorem: If A is an $n \times m$ matrix then the set of solutions to $Ax = 0$ is a subspace of $\mathbb{R}^m$

Proof: a) $A\vec{0} = \vec{0}$ so $\vec{0}$ is in S

b) Suppose $Au = 0$ and $Av = 0$ then $A(u+v) = Au + Av = \vec{0} + \vec{0} = \vec{0}$

c) $A(c\vec{u}) = c(A\vec{u}) = c\vec{0} = \vec{0}$

(If it satisfies all 3 conditions then it is a subspace)